Defects in Skein Theory and the Quantum A-polynomial

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Abstract

We develop the theory of skeins in the presence of codimension one defects. We show that these produce a skein-theoretic model for the quantum decorated character stacks of [JLSS21], thus extending their constructions to three manifolds with surface defects. As an application, we give a quantization of the A-polynomial which refines the well-known construction based on hyperbolic geometry [Dim11].

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1 Introduction

In this paper we initiate the study of skein theory of 3-manifolds with defects, we apply this to construct invariants of knots, more generally of 3-manifolds with boundary, and we develop an effective algebraic formalism to compute the resulting invariants. We apply these ideas is to give a new construction of the so-called "quantum A-polynomial" invariant of knots, in the framework of fully extended TQFT. We apply the formalism to give a succinct and constructive definition of the quantum A-polynomial, and we demonstrate the method by giving computations in many examples. It appears that our computational method is novel even when applied to the classical A-polynomial. We begin the introduction with a brief review of the key concepts, and we move quickly to a demonstration of our main results in the form of a worked example.

The A-polynomial. The classical A-polynomial was constructed in [CCG+94], and has been studied extensively (see for example [CL96, FGL01, Gel01, Gar04, Guk05, Dim11]) . Given a knot K in S^3, one studies a moduli space $\mathcal{M}_K$ of $SL_2(\mathbb{C})$-connections on the knot complement – such moduli spaces are called *character varieties* – whose monodromies along paths connecting boundary points are constrained to lie in the Borel subgroup. Denoting the upper left entry of the meridian and longitude matrix by M and L, the image of $\mathcal{M}_K$ determines a one-dimensional subvariety of $\mathbb{C}^{\times 2}$, and its defining equation $A_K(M, L) = 0$ is called the A-polynomial.

In an influential series of papers from several groups spanning both the mathematics and physics communities [] emerged a proposal for a certain "quantum A-polynomial" knot invariant – an element

$$A_K^q(M, L) \in \mathbb{C}_q[M^{\pm 1}, L^{\pm 1}],$$

which would q-deform the classical A-polynomial.

Among these proposals we highlight (a) a proposal [FGL01] using skein theory, (b) a proposal [?] using canonical quantization and topological recursion, (c) a proposal [] involving the combinatorics of ideal triangulations, and (d) a proposal [?] involving difference equations satisfied by the colored Jones polynomial (sometimes called "the AJ-conjecture"). The four constructions, to our knowledge, are not entirely interrelatable; each has advantages and disadvantages. Our aim is to give a new formalism which refines (a) and (c). In future work, we intend to return to the relation of our work to approach (d).

Defect skein theory. The skein module of a 3-manifold is a vector space spanned by certain labelled graphs embedded in the 3-manifold, modulo local "skein relations", modelled on the graphical calculus of some ribbon tensor category. Two typical examples of ribbon tensor categories of interest are the categories $\mathrm{Rep}_q\, G$ and $\mathrm{Rep}_q\, T$, the categories of integral representations of the quantum groups $U_q\mathfrak{g}$ and $U_q\mathfrak{t}$ respectively; the resulting skein theories are very well-studied, especially in the case $G = \mathrm{SL}_2$, and $T = \mathbb{C}^\times$.

A novel component in this paper is the introduction of what we call *defect skein theory*. Let us fix a 3-manifold M with an embedded surface Σ, and a bipartite coloring of $M \setminus \Sigma$, such that each side of the defect has distinct coloring. The defect skein module consist again of labelled graphs in the 3-manifolds, where the labels and skein relations depend now on a pair of ribbon tensor categories (occupying the "bulk" 3-dimensional regions of M) and a pivotal tensor category (constrained to live on the "defect" Σ).

A typical quantum example of a parabolic defect between the categories $\mathrm{Rep}_q\, G$ and $\mathrm{Rep}_q\, T$ is given by the category $\mathrm{Rep}_q\, B$, of integrable representation of the quantum Borel subalgebra $U_q\mathfrak{b}$.

Decorated character variety. When $q = 1$, the bulk categories become $\mathrm{Rep}\, G$ and $\mathrm{Rep}\, T$, and the defect category becomes $\mathrm{Rep}\, B$. In this case, the skein module we construct has a very natural geometric meaning. In the absence of defects, and with the single bulk region labelled by $\mathrm{Rep}\, G$ the resulting skein module recovers the algebra of functions on the G-character variety of M. []

Given a knot K with knot complement M_K, we may embed a copy of T^2 as the contraction into the interior of M_K. We color the bulk component near the boundary with $\mathrm{Rep}\, T$, the interior component with G, and the boundary with $\mathrm{Rep}\, B$. The defect skein module in this case computes the algebra of functions on the *decorated character variety*, a moduli space parameterising triples consisting of: a G-local system in the G-region, a T-local system in the T-region, and a B-reduction of the product $G \times T$-local system restricted along the surface.

The classical A-polynomial is then naturally reformulated as follows: the T-skein algebra of the boundary torus is naturally isomorphic to the Laurent polynomial algebra $\mathbb{C}[M^{\pm 1}, L^{\pm 1}]$. The annihilator ideal of the empty skein is generated by the classical A-polynomial. Given this observation in the classical case, it is natural to expect – and we indeed rigorously justify – that the quantum deformations we construct give rise to quantisations $A_K^q(M, L) \in \mathbb{C}_q[M^{\pm 1}, L^{\pm 1}]$ of the A-polynomial.

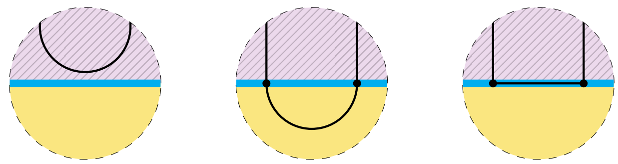

Figure 1: Stratified tangles in $\mathbb{D}_B$ that are not stratified-isotopic.

A worked example: the figure-eight knot Our main results in this paper take the following general form: given any knot K, we define the defect skein module of its knot complement, and we study the resulting action by skeins on the boundary. Given an ideal triangulation of the knot complement, we define an explicit localisation of the defect skein module, and we give a complete computation of this localisation skein module.

In order to give a self-contained exposition of our main results, we now detail the entire computation for the figure-eight knot 4_1, suppressing the machinery which justifies each step in the computation, instead giving pointers to the relevant theorems in the body of the paper. While we focus here on an example knot for expository purposes, we stress however that the method is completely general. Indeed, we link to a github repositiory with code which computes the skein module for any knot.

2 Background

2.1 Stratified spaces

We follow the conventions for stratified spaces used in [AFT17, §2]. A *stratified space* is a paracompact Hausdorff space X together with a continuous map $\varphi : X \to P$ to a poset P, where P is endowed with a topology whose closed sets are generated by $P_{\leq p}$ for $p \in P$. We use the notation $X_p := \varphi^{-1}(p)$ for the individual *strata*.

Remark 2.1. *When* $\dim(X_p) < \dim(X)$*,* X_p *is often referred to as a "defect of codimension* $\dim(X) - \dim(X_p)$*", as a "surface defect" (when* $\dim(X_p) = 2$*) or "line defect" (when* $\dim(X_p) = 1$*) or "point defect" (when* $\dim(X_p) = 0$*). When* $\dim(X) - \dim(X_p) = 1$*,* X_p *is sometimes called an "interface" or "domain wall".*

A map between stratified spaces $X \to P$ and $Y \to Q$ is a pair $F_1 : X \to Y$ and $F_2 : P \to Q$ of continuous maps such that the following square commutes:

$$\begin{array}{ccc} X & \xrightarrow{F_1} & Y \\ \downarrow & & \downarrow \\ P & \xrightarrow{F_2} & Q \end{array} \tag{2.1}$$

We call such a map an embedding if both F_1 and each $F_1|_p : X_p \to Y_{F_2(p)}$ is an embedding. An isotopy between two embeddings F, G of $X \to P$ into $Y \to Q$ is a pair of continuous maps H_1, H_2 such that

$$\begin{array}{ccc} X \times [0,1] & \xrightarrow{H_1} & Y \\ \downarrow & & \downarrow \\ P & \xrightarrow{H_2} & Q \end{array} \tag{2.2}$$

where $X \times [0, 1] \to P$ depends only on the first factor and $H_1(\cdot, 0) = F_1$, $H_1(\cdot, 1) = G_1$. In particular, this implies that $H_2 = F_2 = G_2$. We say that F and G are isotopic if there exists an isotopy between them.

In this paper we will consider surfaces and 3-manifolds stratified over the 3-element poset on letters A, B, C with $A \leq B \geq C$ (so that A and C are not comparable in the poset). We are moreover in the

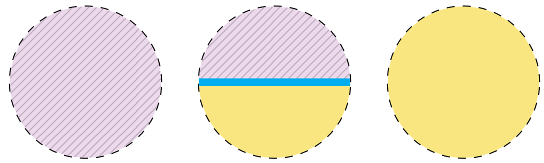

Figure 2: The disks $\mathbb{D}_A$, $\mathbb{D}_B$, and $\mathbb{D}_C$, shown left to right. We will use striped purple, solid teal, and solid yellow to denote A, B, and C regions, respectively.

favorable condition that the strata X_A, X_B, X_C are each smooth submanifolds of co-dimension 0, 1, 0, respectively.

We often denote such spaces as unions of such regions glued along defects, i.e. $X = X_a \cup_{X_b} X_c$ is shorthand for the stratified space $X \xrightarrow{p} \{A \leq B, C \leq B\}$ with $p^{-1}(A) = X_a, p^{-1}(C) = X_c$ both codimension zero and $p^{-1}(B) = X_b$ an embedded surface. Such a defect X_b is sometimes called an *interface* or *domain wall* in the physics literature.

Definition 2.2. *A* **bipartite 3-manifold** *consists of the data of a smooth 3-manifold M, with a continuous map to $\phi : M \to P = (A \leq B \geq C)$, with the property that M_B is a smoothly embedded surface, separating a A-colored and a C-colored 3-dimensional region. We allow 3-manifolds with boundary, and allow the M_B-strata to meet the boundary transversely.*

Definition 2.3. *A* **bipartite surface** *consists of the data of a smooth surface Σ, with a continuous map to $\phi : M \to P = (A \leq B \geq C)$, with the property that M_B is a smoothly embedded curve, separating an open A-colored and an open C-colored 3-dimensional region. We allow surfaces to have boundary, and allow the Σ_B-strata to meet the boundary transversely.*

Inspired by [AFT17, §2.2], we define a *category of basics* whose objects are some specified set of stratified spaces with tangential structure (an orientation in our case) and whose morphisms are smooth open stratified embeddings which respect all tangential structure. A certain subcategory of this will form the local model for the decorated surfaces of our defect skein theory.

Definition 2.4. *Let* $\mathbb{D}\mathrm{isk}$ *denote the (2,1)-subcategory of the category of oriented basics which is generated under disjoint union by the objects $\mathbb{D}_A, \mathbb{D}_B, \mathbb{D}_C$ shown in Figure 2. We will take all three generating objects to have $\{A \leq B, C \leq B\}$ as their underlying poset and consider only those morphisms which respect the label of the strata.*

Remark 2.5. *The poset condition in Definition 2.4 translates simply into the requirement that A-regions can only be embedded into A-regions, C-regions into C-regions, and B-regions into B-regions.*

A subcategory of the basics is a valid source for a disk algebra only if each point in each of its generating objects is contained in some neighborhood isomorphic to another generating object.

Definition 2.6. *The* **(2,1)-category of stratified surfaces**, $\mathbb{S}\mathrm{urf}$ *has:*

objects: *Bipartite surfaces*
1-morphisms *Oriented stratified embeddings.*
2-morphisms: *Stratified isotopies of stratified embeddings.*

Remark 2.7. *We will encounter stratified three dimensional cobordisms between stratified surfaces. Composition of cobordisms is typically defined up to isotopy and a key technical ingredient is the collar neighborhood theorem, which says that the boundary ∂M of a topological/smooth manifold M has a neighborhood diffeomorphic to $(0, 1] \times \partial M$.*

The analogous result does not hold in general for stratified spaces, but holds for stratified 3-manifolds with only smoothly embedded surface defects which meet the boundary transversely.

2.2 Categorical background

Here we recall categorical concepts and definitions encountered in this paper, see [AR94] for complete details.

Definition 2.8. *The 2-category of* **$\mathbb{K}$-linear categories** Cat *has:*

> objects: *Small $\mathbb{K}$-linear categories.*
> 1-morphisms: *$\mathbb{K}$-linear functors.*
> 2-morphisms: *natural transformations*

Definition 2.9. *The 2-category of* **categorical bimodules** Bimod *has:*

> Objects: *Small $\mathbb{K}$-linear categories.*
> 1-morphisms: *From $\mathcal{C}$ to $\mathcal{D}$, these are $\mathbb{K}$-linear functors $F : \mathcal{C} \times \mathcal{D}^{op} \to \mathrm{Vect}$.* [1]
> Composition: *The composition of two bimodules $F : \mathcal{C} \times \mathcal{D}^{op} \to \mathrm{Vect}$ and $G : \mathcal{D} \times \mathcal{E}^{op} \to \mathrm{Vect}$ is given by the coend [ML78, IX.6]:*

$$(F \circ G)(c, e) := \int^{d \in \mathcal{D}} F(c, d) \otimes G(d, e) \tag{2.3}$$

> 2-morphisms: *Natural transformations.*

One obtains a fully faithful 2-functor Cat $\to$ Bimod, which is the identity on objects, which sends a functor $F : \mathcal{C} \to \mathcal{D}$ to the bimodule

$$\begin{aligned} \mathcal{C} \times \mathcal{D}^{op} &\to \mathrm{Vect} \\ (c, d) &\mapsto \mathrm{Hom}(d, F(c)), \end{aligned} \tag{2.4}$$

and which at the level of 2-morphisms maps a natural transformation to a bimodule homomorphisms in the obvious way.

Definition 2.10. *The 2-category of* **locally presentable categories** Pr^L *has:*

> Objects: *Locally presentable $\mathbb{K}$-linear categories.*
> 1-morphisms: *cocontinuous functors.*
> 2-morphisms: *natural transformations.*

Let $\mathrm{Vect} \in \mathrm{Pr}^L$ denote the category of vector spaces (whose basis may be of arbitrary cardinality). Given $\mathcal{C} \in \mathrm{Cat}$, we will denote by $\widehat{\mathcal{C}} := \mathrm{Fun}(\mathcal{C}^{op}, \mathrm{Vect}) \in \mathrm{Pr}^L$ the associated category of presheaves. A category in Pr^L which arises as the free cocompletion of a small category $\mathcal{C}$ is said to "have enough compact projectives". Indeed, every object of

We note that $\mathcal{C}$ maps functorially to its free cocompletion via the Yoneda embedding, $\mathcal{C} \to \widehat{\mathcal{C}}$, $c \mapsto \widehat{c} := \mathrm{Hom}(-, c)$. The category $\widehat{\mathcal{C}}$ is sometimes called the *free cocompletion* of $\mathcal{C}$ because it satisfies the usual universal property for cocontinuous functors out of it.

One has the further structure of a 2-functor $\widehat{\cdot} : \mathrm{Bimod} \to \mathrm{Pr}^L$, also called free cocompletion as follows. The free cocompletion $\widehat{N}$ of a bimodule $N : \mathcal{C} \times \mathcal{D}^{op} \to \mathrm{Vect}$ is the cocontinuous functor $\widehat{N} : \widehat{\mathcal{C}} \to \widehat{\mathcal{D}}$,

$$\widehat{N}(S) : d \mapsto \int^{c \in \mathcal{C}} N(c, d) \otimes S(c). \tag{2.5}$$

In the special case that N is obtained from a functor F, the free cocompletion $\widehat{F}$ of the functor $F : \mathcal{C} \to \mathcal{D}$ is given by the coend

$$\hat{F}(S)(d) = \int^{c} \mathrm{Hom}(d, F(c)) \otimes S(c). \tag{2.6}$$

A functor between categories in Pr^L with enough compact projectives lies in the image of the cocompletion functor if, and only if, it preserves compact projectives objects, or equivalently if, and only if, its right adjoint is itself cocontinuous.

[1] Such functors are sometimes called *bimodules* or *profunctors*.

3 Defect skein theory

A central part of this work is to define skein theory of a bipartite surfaces and three-manifolds, as a (3,2)-TFT with defects, following [Wal06] and [JF21] and [RT90] in the unstratified case. Our constructions will involve a pair of ribbon tensor categories $\mathcal{A}$ and $\mathcal{C}$ assigned to the open A- and C- regions, respectively, together with the data of a pivotal tensor category $\mathcal{B}$ assigned to the defect. There is an extra structure relating $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$, captured in the following definition.

Definition 3.1. *Fix ribbon tensor categories $\mathcal{A}$ and $\mathcal{C}$. A pivotal $(\mathcal{A}, \mathcal{C})$-central tensor category is a pivotal tensor category $\mathcal{B}$ together with a pivotal braided tensor functor $\mathcal{A} \boxtimes \overline{\mathcal{C}} \to \mathcal{Z}(B)$.*

Here $\mathcal{Z}(\mathcal{B})$ denotes the Drinfeld center of $\mathcal{B}$ equipped with its canonical pivotal braided tensor structure, while $\overline{\mathcal{C}}$ denotes the tensor category $\mathcal{C}$ equipped with the opposite braiding and inverse ribbon structure. We will abbreviate simply by $\mathcal{B}$ the data

$$\mathcal{B} := \big(\mathcal{A}, \mathcal{B}, \mathcal{C}, (H, J) : \mathcal{A} \boxtimes \overline{\mathcal{C}} \to Z(\mathcal{B})\big). \tag{3.1}$$

Following [BJS21, Figure 2] and [FSV13], we will use the central structure to determine how skeins are 'pushed' from the bulk into the defect. We package this information as a functor analogous to the aforementioned braided tensor functors $\mathrm{RT}_{\mathcal{A}} : \mathrm{Rib}_{\mathcal{A}} \to \mathcal{A}$ and $\mathrm{RT}_{\mathcal{C}} : \mathrm{Rib}_{\mathcal{C}} \to \mathcal{C}$ introduced in [RT90].

3.1 Defect skeins and pivotal central tensor categories

The aim of this section is to spell out a skein theory suitable to bipartite surfaces and 3-manifolds.

Definition 3.2. *A $\mathcal{B}$-labelling of a bipartite surface is a finite collection $\{(x_i, V_i, \epsilon_i)\}$ of signed (denoted $\epsilon_i \in \{\pm 1\}$), framed points x_i, each one labelled with an object V_i, of $\mathcal{A}$, $\mathcal{B}$, or $\mathcal{C}$, according to which region the point x_i occupies.*

Definition 3.3. *A stratified ribbon graph in M is an embedded ribbon graph in M, together with...*

Definition 3.4. *The category $\mathrm{Rib}_{\mathcal{B}}$ of colored ribbon graphs near the defect has:*

- Objects: *$\mathcal{B}$-labelings of $\mathbb{D}_B$. See Figure 5.*
- Morphisms: *Oriented and stratified colored ribbon graphs embedded in $\mathbb{D}_B \times [0, 1]$, compatible with a $\mathcal{B}$-labeling of $\mathbb{D}_B \times \{0, 1\}$. See Figure 4.*
- Monoidal Structure *induced by a embedding $\mathbb{D}_B \sqcup \mathbb{D}_B \hookrightarrow \mathbb{D}_B$. The unit $\mathbb{1}$ is the empty object.*
- Duality: *Fix an object X in $\mathrm{Rib}_{\mathcal{B}}$. Its dual X^* is obtained by reversing the orientation, i.e. reversing the signs on all marked points. Coevaluation $\mathbb{1} \to X \otimes X^*$ is given by connecting each marked point with its partner by an arc that remains in a single stratum. We can ensure the arcs are not entangled, e.g. by representing them by flat half-circles. Evaluation $X^* \otimes X \to \mathbb{1}$ is described similarly.*

For the coefficient system considered in Section 5, the pivotal category assigned to the defect is generated under co-limits by the image of the central structure. This means the hom-spaces in our skein categories will be generated by diagrams that are at most transverse to the defect. This motivates the following definition.

Definition 3.5. *The subcategory $\mathrm{Rib}_{\mathcal{B}}^{\pitchfork} \subset \mathrm{Rib}_{\mathcal{B}}$ has as its objects those $\mathcal{B}$-labelings none of whose points x_i lie on the B-defect, and as its morphisms only those ribbon graphs whose B-labelled componets are intervals.*

Remark 3.6. *As a caution we note that stratified isotopy is a weaker equivalence relation than isotopy relative to the defect, since the former allows the location where a strand meets a defect to change. For example, the tangles in Figure 6 are stratified-isotopic but not isotopic relative to the defect. A coupon in the defect may have edges transverse to the surface, see Figure 4. In this case we implicitly pre-compose with the central structure and think of the underlying morphism as being between objects in $\mathcal{B}$.*

Remark 3.7. *Any stratified embedding $\sqcup_i \mathbb{D}_{R_i} \hookrightarrow \sqcup_j \mathbb{D}_{S_j}$ between finite collections of disks induces a functor between the Deligne-Kelly tensor products of the associated categories of ribbons. This is because colored ribbon graphs are themselves stratified embeddings and so the pullback of the disk embeddings are again morphisms. Similarly, an isotopy between two embeddings induces a natural transformation in the category of ribbon graphs. We will make frequent use of such induced functors throughout the paper.*

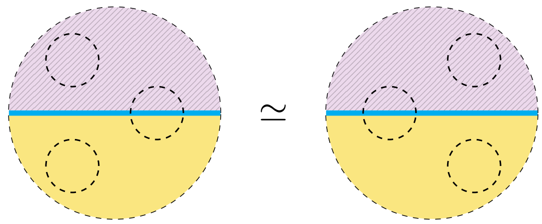

Figure 3: An isotopy induces the half braiding that gives $\mathrm{Rib}_{\mathcal{B}}$ a $(\mathrm{Rib}_{\mathcal{A}}, \mathrm{Rib}_{\mathcal{C}})$-central structure.

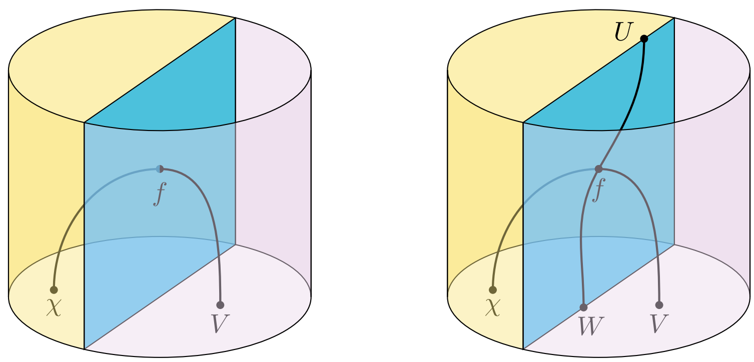

Figure 4: Stratified colored tangles passing through a surface defect. At left, the is transverse to the defect.

Definition-Proposition 3.8. $\mathrm{Rib}_{\mathcal{B}}$ *is a pivotal category and a pivotal* $(\mathrm{Rib}_{\mathcal{A}}, \mathrm{Rib}_{\mathcal{C}})$*-central algebra.*

Proof. The dual of an object $X \in \mathrm{Rib}_{\mathcal{B}}$ is obtained by switching the sign on all points. The canonical isomorphism $i_X : X^{**} \xrightarrow{\simeq} X$ is a monoidal natural isomorphism, since changing the orientation of marked points commutes with the monoidal structure.

The central structure consists of a braided tensor functor, $H^{rib} : \mathrm{Rib}_{\mathcal{A}} \boxtimes \overline{\mathrm{Rib}_{\mathcal{C}}} \to \mathrm{Rib}_{\mathcal{B}}$ together with a half braiding J^{rib}. Following [BJS21, Fig. 2], this is constructed by lifting the functor induced by the embedding $\mathbb{D}_A \sqcup \overline{\mathbb{D}}_C \hookrightarrow \mathbb{D}_B$ to the Drinfeld center. The half-braiding is induced by an isotopy between the two embeddings $\mathbb{D}_A \sqcup \mathbb{D}_B \sqcup \mathbb{D}_C \hookrightarrow \mathbb{D}_C$ shown in Figure 3.

□

Definition-Proposition 3.9. *The following procedure determines an essentially surjective, full, pivotal monoidal functor* $\mathrm{RT}_{\mathcal{B}} : \mathrm{Rib}_{\mathcal{B}} \to \mathcal{B}$*:*

Given a defect ribbon graph T in $\mathbb{D}_B \times [0,1]$, first pick a representative in generic position with respect to the orthogonal projection onto the defect, so that all crossings are transverse and coupons are not identified by the projection. Project this representative onto the defect, replacing all crossings with a coupon labelled by the appropriate half-crossing in $Z(\mathcal{B})$.Extend linearly to finite sums of defect tangles in $\mathbb{D}_B \times [0,1]$

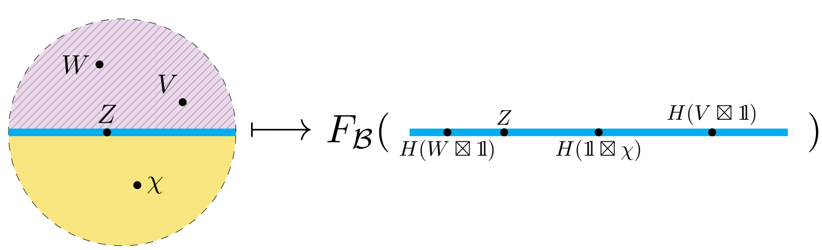

Figure 5: The functor $\mathrm{RT}_{\mathcal{B}}$ on objects of $\mathrm{Rib}_{\mathcal{B}}$. Here H comes from the central structure $\mathcal{A} \boxtimes \overline{\mathcal{C}} \to Z(\mathcal{B})$.

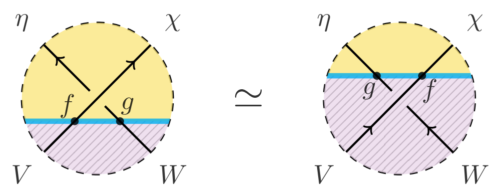

Figure 6: A birds-eye view of isotopic tangles, demonstrating that crossings can move through defects. The coupons can move past each other because the isotopy takes place in a thickening of the disk.

Next use the planar diagram evaluation functor $F_{\mathcal{B}}$ to get a morphism in $\mathcal{B}$.

Proof. We start by showing $\mathrm{RT}_{\mathcal{B}}$ is well defined, i.e. it respects isotopies of the underlying stratified ribbon graphs and doesn't depend on the projection of the bulk ribbon graphs into the defect. It suffices to show that it is invariant under stratified versions of the Reidemeister moves. Let $H : \mathcal{A} \boxtimes \overline{\mathcal{C}} \to \mathcal{B}$ be the braided tensor functor of the central structure composed with the forgetful functor $Z(\mathcal{B}) \to \mathcal{B}$ and $J_{a,c,b} : H(a \boxtimes c) \otimes b \xrightarrow{\simeq} b \otimes H(a \boxtimes c)$ the half-braiding.

Crossings can pass through defects: We show that the isotopic graphs in Figure 6 are sent to the same morphism in $\mathcal{B}$. Let $f : H(V \boxtimes \mathbb{1}_{\mathcal{C}}) \to H(\mathbb{1}_{\mathcal{A}} \boxtimes \chi)$ and $g : H(W \boxtimes \mathbb{1}_{\mathcal{C}}) \to H(\mathbb{1}_{\mathcal{A}} \boxtimes \eta)$ be the labels for the two coupons in the tangles shown in Figure 6. Then $\mathrm{RT}_{\mathcal{B}}$ maps the left tangle of Figure 6 to the morphism

$$H(V \boxtimes \mathbb{1}) \otimes H(W \boxtimes \mathbb{1}) \xrightarrow{f \otimes g} H(\mathbb{1} \boxtimes \chi) \otimes H(\mathbb{1} \boxtimes \eta) \simeq H(\mathbb{1} \boxtimes \chi \otimes \eta) \xrightarrow{H\left(\mathrm{id} \boxtimes \sigma^{\overline{\mathcal{C}}}\right)} H(\mathbb{1} \boxtimes \eta \otimes \chi) \tag{3.2}$$

While the tangle on the right is sent to

$$H(V \otimes W \boxtimes \mathbb{1}) \xrightarrow{H(\sigma^{\mathcal{A}} \boxtimes \mathrm{id})} H(W \otimes V \boxtimes \mathbb{1}) \simeq H(W \boxtimes \mathbb{1}) \otimes H(V \boxtimes \mathbb{1}) \xrightarrow{g \otimes f} H(\mathbb{1} \boxtimes \eta) \otimes H(\mathbb{1} \boxtimes \chi) \tag{3.3}$$

By assumption H is a braided tensor functor, so $\sigma^{Z(\mathcal{B})} = H\left(\sigma^{\mathcal{A} \boxtimes \overline{\mathcal{C}}}\right) = H\left(\sigma^{\mathcal{A}} \boxtimes \sigma^{\overline{\mathcal{C}}}\right)$. It follows that

$$\sigma^{Z(\mathcal{B})}|_{H(\mathbb{1} \boxtimes -)} = H\left(\mathrm{id} \boxtimes \sigma^{\overline{\mathcal{C}}}\right) \quad \text{and} \quad \sigma^{Z(\mathcal{B})}|_{H(- \boxtimes \mathbb{1})} = H(\sigma^{\mathcal{A}} \boxtimes \mathrm{id}). \tag{3.4}$$

Since the braiding is a natural transformation, we conclude

$$\begin{aligned} H\left(\mathrm{id} \boxtimes \sigma^{\overline{\mathcal{C}}}\right) \circ (f \otimes g) &= \sigma^{Z(\mathcal{B})} \circ (f \otimes g) \\ &= (g \otimes f) \circ \sigma^{Z(\mathcal{B})} \\ &= (g \otimes f) \circ H\left(\sigma^{\mathcal{A}} \boxtimes \mathrm{id}\right). \end{aligned}$$

It follows that (3.2) and (3.3) are the same morphism up to composition with the isomorphisms $H(- \otimes -) \simeq H(-) \otimes H(-)$.

Stratified Reidemeister 3 holds: Our previous result means we only need to consider tangles where no strand moves between regions. Even so, the three strands can be in some combination of the $\mathcal{A}, \mathcal{B}$, and $\mathcal{C}$ regions.

The functor $\mathrm{RT}_{\mathcal{B}}$ sends any crossing to the appropriate braiding in $Z(\mathcal{B})$, so the various stratified versions of the Yang-Baxter equation all simplify to those of $Z(\mathcal{B})$, where some of the objects happen to be in the image of the functor H.

Stratified Reidemeister 2 holds: By the same argument as above, this reduces to and hence follows from the analogous statement about the invertibility of the braiding in $Z(\mathcal{B})$.

Both fullness and essential surjectivity follow from Lemma **??**, which shows that $F_{\mathcal{B}} : \mathrm{Planar}_{\mathcal{B}} \to \mathcal{B}$ is both. □

Definition 3.10. *Let* $\mathrm{RT}_{\mathcal{B}} : \mathrm{Rib}_{\mathcal{B}} \to \mathcal{B}$ *be the functor described in Lemma 3.9.*

Lemma 3.11. *Skein relations are compatible with embeddings. Let* $\iota_A : \mathbb{D}_A \hookrightarrow \mathbb{D}_B$ *and* $\iota_C : \mathbb{D}_C \hookrightarrow \mathbb{D}_B$ *be stratified embeddings. If two* $\mathcal{A}$ *(resp.* $\mathcal{C}$*) colored ribbon graphs are equivalent as* $\mathcal{A}$ *(resp.* $\mathcal{C}$*) skeins, then their push forwards along* ι_A *(resp.* ι_C*) are equivalent as defect skeins.*

Proof. The claim is that the following diagrams commute:

$$\begin{array}{ccc} \mathrm{Rib}_{\mathcal{A}} & \xrightarrow{(\iota_A)_*} & \mathrm{Rib}_{\mathcal{B}} \\ \downarrow{\scriptstyle \mathrm{RT}_{\mathcal{A}}} & & \downarrow{\scriptstyle \mathrm{RT}_{\mathcal{B}}} \\ \mathcal{A} \simeq \mathcal{A} \boxtimes \mathbb{1} & \xrightarrow{H|_{\mathcal{A}\boxtimes \mathbb{1}}} & \mathcal{B} \end{array} \qquad \begin{array}{ccc} \mathrm{Rib}_{\mathcal{C}} & \xrightarrow{(\iota_C)_*} & \mathrm{Rib}_{\mathcal{B}} \\ \downarrow{\scriptstyle \mathrm{RT}_{\mathcal{C}}} & & \downarrow{\scriptstyle \mathrm{RT}_{\mathcal{B}}} \\ \mathcal{C} \simeq \mathbb{1} \boxtimes \mathcal{C} & \xrightarrow{H|_{\mathbb{1}\boxtimes \mathcal{C}}} & \mathcal{B} \end{array} \tag{3.5}$$

□

3.2 The defect skein category

We now define the defect skein category and associated constructions analogously to the unstratified skein theory, using our defect Reshetikhin-Turaev evaluation functors as a local model near the domain walls. Fix a compact oriented 3-manifold, possibly with boundary, and a $\mathcal{B}$-labeling X on ∂M.

Definition 3.12. *The* **relative defect skein module** $\mathrm{Sk}_{\mathcal{B}}(M, X)$ *is the* $\mathbb{K}$*-module spanned by isotopy classes of stratified ribbon graphs in* M *compatible with* X*. These are taken modulo the relations induced by* $\mathrm{RT}_{\mathcal{A}}, \mathrm{RT}_{\mathcal{C}}$*, and* $\mathrm{RT}_{\mathcal{B}}$ *from the embedding of any cylinder* $\mathbb{D}_{\mathcal{R}} \times [0,1] \hookrightarrow M$ *which respects the stratification and labeling of* M*. Here* $\mathcal{R} = \mathcal{A}, \mathcal{B}, \mathcal{C}$*.*

As a special case $X = \emptyset$, we abbreviate by $\mathrm{Sk}_{\mathcal{B}}(M)$ and refer to this as the **defect skein module** of M.

Definition 3.13. *The* **defect skein category** $\mathrm{SkCat}_{\mathcal{B}}(\Sigma)$ *has*

objects *Finite collections of oriented framed marked points in* Σ*, colored by objects of the category associated to their region.*

morphisms: *The homomorphism space from* X *to* Y *is the relative defect skein module*

$$\mathrm{Hom}(X, Y) := \mathrm{Sk}_{\mathcal{B}}(\Sigma \times [0,1], \overline{X} \sqcup Y), \tag{3.6}$$

where $\overline{X}$ *is on* $\Sigma \times \{0\}$ *and* Y *is on* $\Sigma \times \{1\}$*. Composition is given by stacking copies of the thickened surface and a smoothing at boundary points.*

We note that $\mathrm{SkCat}_{\mathcal{B}}(\mathbb{D}_A) \simeq \mathcal{A}$ and $\mathrm{SkCat}_{\mathcal{B}}(\mathbb{D}_C) \simeq \mathcal{C}$ as ribbon categories. Now we show a similar result for $\mathrm{SkCat}_{\mathcal{B}}(\mathbb{D}_B)$.

Lemma 3.14. *We have an equivalence* $\mathrm{SkCat}_{\mathcal{B}}(\mathbb{D}_B) \simeq \mathcal{B}$ *as pivotal* $(\mathcal{A}, \mathcal{C})$*-central algebras.*

Proof. By Lemma 3.9, the pivotal functor $\mathrm{RT}_{\mathcal{B}} : \mathrm{Rib}_{\mathcal{B}} \to \mathcal{B}$ is full and essentially surjective. Since skein relations are exactly the kernel of this map, it follows that $\mathrm{RT}_{\mathcal{B}}$ induces an equivalence of pivotal categories $\mathrm{SkCat}_{\mathcal{B}}(\mathbb{D}_B) \simeq \mathcal{B}$.

By Remark 3.7, the $(\mathrm{Rib}_A, \mathrm{Rib}_C)$-central structure on $\mathrm{Rib}_{\mathcal{B}}$ descends to the skein category. Let H^{Sk}, J^{Sk} be the central structure on $\mathrm{SkCat}_{\mathcal{B}}(\mathbb{D}_B)$. Equivalence of the central structures can be expressed by this commutative diagram:

$$\begin{array}{ccc} \mathrm{SkCat}_{\mathcal{B}}(\mathbb{D}_A) \boxtimes \mathrm{SkCat}_{\mathcal{B}}(\overline{\mathbb{D}}_C) & \xrightarrow{H^{\mathrm{Sk}}, J^{\mathrm{Sk}}} & Z(\mathrm{SkCat}_{\mathcal{B}}(\mathbb{D}_B)) \\ \downarrow{\scriptstyle \mathrm{RT}_{\mathcal{A}} \boxtimes \mathrm{RT}_{\overline{\mathcal{C}}}} & & \downarrow{\scriptstyle Z(\mathrm{RT}_{\mathcal{B}})} \\ \mathcal{A} \boxtimes \overline{\mathcal{C}} & \xrightarrow{H, J} & Z(\mathcal{B}) \end{array}$$

Note that a functor $F : \mathcal{C} \to \mathcal{D}$ doesn't necessarily induce one between Drinfeld centers. To get the right-most vertical arrow in the above diagram we've used that $\mathrm{RT}_{\mathcal{B}}$ is a categorical equivalence. Tangles entirely in the bulk regions of $\mathbb{D}_B$ are exactly those in the image of H^{Sk}. In the construction of $\mathrm{RT}_{\mathcal{B}}$ such tangles are projected onto the defect where H gives the appropriate objects and morphisms of $\mathcal{B}$. This implies that $\mathrm{RT}_{\mathcal{B}} \circ H^{\mathrm{Sk}} = H \circ (\mathrm{RT}_{\mathcal{A}} \boxtimes \mathrm{RT}_{\mathcal{C}})$.

When projection onto the defect introduces crossings, they are replaced by coupons labeled by the half braiding J. The half braiding J^{Sk} is induced by the isotopy in Figure 3, which introduces crossings between strands. Commutativity of the diagram follows from the fact that $\mathrm{RT}_{\mathcal{B}}$ sends a coupon to the labelling morphism. □

Definition 3.15. *The transverse defect skein category* $\mathrm{SkCat}^{\pitchfork}_{\mathcal{B}}(\Sigma)$ *is the subcategory ...*

Corollary 3.16. *We have an equivalence* $\mathrm{SkCat}^{\pitchfork}_{\mathcal{B}}(\Sigma) \simeq \underline{\mathrm{End}}(\mathbf{1}_{\mathcal{B}})\mathrm{mod}_{\mathcal{A}\boxtimes\overline{\mathcal{C}}}$

4 Monadic reconstruction for defect skeins

Skein categories as defined in the preceding section are of limited computational utility due to their non-finitary definition: one has as an infinite-dimensional space of possible skeins, modulo an even more infinite set of relations. An algebraic approach to skein theory was initiated in [BZBJ18] and pursued in [?]. This involves using monadic reconstruction to describe the free completions of skein categories of surfaces as categories of modules for certain finitely presented algebras called internal skein algebras. This was extended in [?] to describe the functors given by cobordisms between surfaces via internal skein bimodules over internal skein algebras.

In this section, we extend the monadic formalism to the defect setting, noting a few key differences from the unstratified setting.

In what follows $(\mathcal{A}, \mathcal{C}, (H, J) : \mathcal{A} \boxtimes \overline{\mathcal{C}} \to Z(\mathcal{B}))$ denotes a pair of ribbon categories along with a pivotal category and $(\mathcal{A}, \mathcal{C})$-central algebra $\mathcal{B}$. We use H to mean the functor $\mathcal{A} \boxtimes \overline{\mathcal{C}} \to \mathcal{B}$ induced by forgetting the half braiding on the Drinfeld center and J to mean the half braiding.

The *internal skein algebra* of a surface is defined via disk insertion at a distinguished interval, called a gate, along a boundary component. The location of the gate has minimal impact for path-connected surfaces. Adding additional gates in a connected component induces a categorical equivalence on the category of internal modules for the internal defect skein algebra, [JLSS21, Lemma 4.1]. In the presence of defects, we must instead consider connected components of the top dimensional strata. Therefore, in general, we will work with many gates on a single stratified surface. See [Haï22, §7.2] for a careful treatment (in the unstratified setting) of internal skein algebras defined with multiple gates.

Besides this mild difference, we follow the definition of internal defect skein algebras and modules given in [GJS21]. We assume that our surfaces have non-empty boundary, see Section 4.2.1 on puncturing surfaces.

Definition 4.1. *Let* $\Sigma \simeq \Sigma_A \cup_{\Gamma_B} \Sigma_C$ *denote a stratified surface and* $\mathcal{G}$ *be a collection of disjoint intervals (called* **gates***) along* $\partial\Sigma$*, also disjoint from the interface* Γ_B*. The inclusion* $\mathcal{G} \times [0,1] \hookrightarrow \Sigma$ *induces the* **disk insertion functor***:*

$$\mathcal{P} : \mathrm{SkCat}(\mathcal{G} \times [0,1]) \to \mathrm{SkCat}(\Sigma) \tag{4.1}$$

Suppose there are n *gates in the* A *region, and* m *in the* C *region. Then* $\mathrm{SkCat}(\mathcal{G} \times [0,1]) \simeq \mathcal{A}^{\boxtimes n} \boxtimes \mathcal{C}^{\boxtimes m}$*. We frequently consider* $\mathcal{P}$ *as a functor from the product category. Although it introduces conflicting notation, we will use the shorthand* $\mathcal{G} := \mathcal{A}^{\boxtimes n} \boxtimes \mathcal{C}^{\boxtimes m}$*.*

In the free cocompletion, the functor $\mathcal{P}$ induced by disk insertion is guaranteed to have a right adjoint:

$$\widehat{\mathcal{P}}^R : \widehat{\mathrm{SkCat}}(\Sigma) \to \widehat{\mathrm{SkCat}}(\mathcal{G} \times [0,1]). \tag{4.2}$$

The monad $\widehat{\mathcal{P}}^R \widehat{\mathcal{P}}$ of this adjuction is an endofunctor of $\widehat{\mathrm{SkCat}}(\mathcal{G} \times [0,1]) \simeq \mathrm{Fun}\big(\mathcal{A}^{\boxtimes n} \boxtimes \mathcal{C}^{\boxtimes m}, \mathrm{Vect}\,\big)$.

Definition 4.2. *Evaluating the monad induced by disk insertion at the unit* $\widehat{\mathbb{1}} = \mathrm{Hom}(\mathbb{1}, -)$ *gives an algebra object in the free cocompletion of* $\mathcal{G} = \mathcal{A}^{\boxtimes n} \boxtimes \mathcal{C}^{\boxtimes m}$*. We will call it the* **defect internal skein algebra***:*

$$\mathrm{SkAlg}^{int}_{\mathcal{G}}(\Sigma) := \widehat{\mathcal{P}}^R \widehat{\mathcal{P}}(\widehat{\mathbb{1}}). \tag{4.3}$$

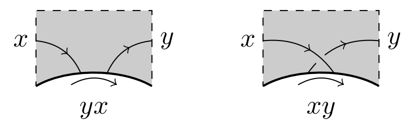

Figure 7: The incident edges in a product are ordered right to left according to the orientation at a gate.

Note that $\mathcal{G} \times [0,1]$ is a thickened curve and therefore its skein category has a monoidal structure. This is the source of the product on the internal skein algebra. In [GJS21], the internal skein algebra is equivalently defined as a lax monoidal functor

$$\begin{aligned} \mathrm{SkAlg}^{\mathrm{int}}_{\mathcal{G}}(\Sigma) : \mathcal{G} &\to \mathrm{Vect} \\ V &\mapsto \mathrm{Sk}\left(\Sigma \times [0,1]; \mathcal{P}(V), \varnothing\right). \end{aligned} \tag{4.4}$$

The equivalence of these definitions follows from the fact that an algebra object in $\widehat{\mathcal{G}}$ is exactly a lax monoidal functor $\mathcal{G} \to \mathrm{Vect}$, after the short computation:

$$\begin{aligned} \mathrm{SkAlg}^{int}_{\mathcal{G}}(\Sigma)(V) &\simeq \mathrm{Hom}_{\widehat{\mathcal{G}}}\left(\widehat{V}, \widehat{\mathcal{P}}^R \widehat{\mathcal{P}}(\widehat{\mathbb{1}})\right) \simeq \mathrm{Hom}_{\widehat{\mathrm{SkCat}(\Sigma)}}\left(\widehat{\mathcal{P}(V)}, \widehat{\mathcal{P}(\mathbb{1})}\right) \\ &\simeq \mathrm{Hom}_{\mathrm{SkCat}(\Sigma)}\left(\mathcal{P}(V), \mathcal{P}(\mathbb{1})\right) = \mathrm{Sk}(\Sigma \times [0,1]; \mathcal{P}(V)). \end{aligned} \tag{4.5}$$

Here the first and third equivalence use the Yoneda lemma while the second follows from the definition of an adjoint together with $\widehat{\mathcal{P}}(\widehat{W}) \simeq \widehat{\mathcal{P}(W)}$.

Definition 4.3. *Let $M = M_A \cup_{S_B} M_C$ be a stratified 3-manifold. Fix an identification $\partial M = \Sigma \cup_\Gamma R$. For the purposed of disk insertion we treat Γ as the boundary of ∂M. Fix gates $\mathcal{G} \times [0,1] \hookrightarrow \Sigma$ with associated disk insertion functor $\mathcal{P}$. Recall the shorthand $\mathcal{G} := \mathrm{SkCat}(\mathcal{G} \times [0,1])$. The* **defect internal skein module of** *M is the functor*

$$\mathrm{Sk}^{int}(M) : \mathcal{G} \to \mathrm{Vect} \tag{4.6}$$

given by

$$V \mapsto \mathrm{Sk}\left(M; \mathcal{P}(V)\right). \tag{4.7}$$

This is a $\mathrm{SkAlg}^{int}(\Sigma)$ module internal to the free cocompletion of $\mathcal{G}$, meaning there is a morphism of presheaves $\mathrm{SkAlg}^{int}(\Sigma) \otimes \mathrm{Sk}^{int}(M) \to \mathrm{Sk}^{int}(M)$. By (4.5), $\mathrm{SkAlg}^{int}(\Sigma) \simeq \mathrm{Sk}^{int}(\Sigma \times [0,1])$ as presheaves.

We take a moment to give a more concrete description of the internal skein algebra and module. First, by the co-Yoneda lemma

$$\mathrm{Sk}^{int}(M) \simeq \int^{V \in \mathcal{G}} \widehat{V} \otimes \mathrm{Sk}(M, \mathcal{P}(V)). \tag{4.8}$$

Hence an element of the internal skein module is a *state* $v \in V$ together with a skein in M whose restriction to ∂M is the labelling $\mathcal{P}(V)$, up to the coend equivalence relation. Because labellings $\mathcal{P}(V)$ have marked points only immediately adjacent to gates, we will draw skeins as starting and ending on the gates themselves. When multiple edges of a skein meet at a single gate in a thickened surface, their relative heights are recorded pictorial according to the orientation of the boundary component. See Figure 7.

Remark 4.4. *Hence, internal skein modules and defect internal skein modules are, up to fixing conventions, equivalent to so-called "stated skeins" constructions [**?**]. In particular, in the presence of parabolic induction defects, one lands on definitions very close to that of Müller's early work.*

4.1 Cobordisms and gluing

We now define a categorified oriented 2+1 TQFT for cobordisms with smoothly embedded codimension-one defects. An important ingredient for gluing is that the boundaries of our cobordisms admit stratified collared neighborhoods, see Remark 2.7. For our main application (see Section 7) we will need to understand stratified analogs of one and two handle attachments and three dimensions.

Definition 4.5. *Let M be a stratified 3-manifold as in Definition 3.12, with the additional data of a decomposition of its boundary $\partial M \simeq (\overline{\Sigma}_{in} \sqcup \Sigma_{out}) \cup R$. We will write $\underline{\mathrm{Sk}}(M) : \mathrm{SkCat}(\Sigma_{in})^{op} \times \mathrm{SkCat}(\Sigma_{out}) \to \mathrm{Vect}$ for the bimodule defined by the relative defect skein module $X_{in}, X_{out} \mapsto \mathrm{Sk}_{\mathcal{B}}(M, \overline{X}_{in} \sqcup X_{out})$, and refer to this as the* **defect skein bimodule functor**.

The following result is a mild generalization of [GJS21, Theorem 2.5] and [Wal06, Theorem 4.4.2] to oriented 3-manifolds with smoothly embedded codimension-one strata.

Lemma 4.6. *Let M be an oriented stratified 3-manifold with a stratification $M \simeq M_A \cup_{S_B} M_C$. Fix a decomposition $\partial M \simeq \Sigma_{gl} \cup \overline{\Sigma}_{gl} \cup R$ and let M_{gl} denote M with the two copies of Σ_{gl} identified. There is an equivalence of functors $\mathrm{SkCat}(R)^{op} \to \mathrm{Vect}$*

$$\underline{\mathrm{Sk}}(M_{gl})(-) \simeq \int^{X \in \mathrm{SkCat}(\Sigma_{gl})} \underline{\mathrm{Sk}}(M)(\overline{X} \sqcup X \sqcup -). \tag{4.9}$$

The coend in (4.9) motivates our earlier requirement that the coefficient categories $\mathcal{A}, \mathcal{B}, \mathcal{C}$ are themselves small. The proof of [Wal06, Theorem 4.4.2] applies here after checking that claims regarding isotopies can be made about stratified isotopies. The main claim is that collar shift isotopies and stratified isotopies supported in balls disjoint from $\Sigma_{gl} \sqcup \overline{\Sigma}_{gl}$ generate stratified isotopies on M_{gl}.

Let $\Sigma_1 \overset{M_{12}}{\to} \Sigma_2 \overset{M_{23}}{\to} \Sigma_3$ be a pair of composable stratified cobordisms. Applying Lemma 4.6 to the case when $M = M_{12} \sqcup M_{23}$, $\Sigma_{gl} = \Sigma_2$, and $R = \Sigma_1 \sqcup \Sigma_3$, we obtain

$$\mathrm{Sk}(M_{12} \cup_{\Sigma_2} M_{23}; \overline{X}_1 \sqcup X_3) \simeq \int^{X_2 \in \mathrm{SkCat}(\Sigma_2)} \mathrm{Sk}(M_{12}; \overline{X}_1 \sqcup X_2) \otimes \mathrm{Sk}(M_{23}; \overline{X}_2 \sqcup X_3) \tag{4.10}$$

naturally in $X_1 \in \mathrm{SkCat}(\Sigma_1)$ and $X_3 \in \mathrm{SkCat}(\Sigma_3)$ This gives a useful corollary:

Corollary 4.7. *Together $\mathrm{SkCat}_{\mathcal{B}}(-)$ and $\underline{\mathrm{Sk}}(-)$ define a contravariant functor from the category of oriented 2+1 collared cobordisms with smoothly embedded codimension-one defects to* Bimod. *We will denote this categorified 2+1 TQFT by*

$$\underline{\mathrm{Sk}}_{\mathcal{B}} : \mathbb{C}\mathrm{ob}_{2+1}^{or} \to \mathrm{Bimod} \tag{4.11}$$

We rely heavily on Lemma 4.6 in Section 7.2, where we use the gluing properties of defect skein categories to decompose 3-manifolds into triangulations.

4.2 Cobordisms and gluing for internal skein algebras and modules

We now consider gluing for internal skein modules and algebras. The main application we have in mind is gluing together tetrahedra in a triangulated three-manifold. See Section **??** for a detailed description in the case of decorated skein algebras.

Lemma 4.8. *Let M be an oriented stratified 3-manifold with a stratification $M \simeq M_A \cup_{S_B} M_C$. Fix a decomposition $\partial M \simeq \Sigma_{gl} \cup \overline{\Sigma}_{gl} \cup R$ and let M_{gl} denote M with the two copies of Σ_{gl} identified. Let $\mathcal{G}_{gl}$ denote the gate category on Σ_{gl} and $\mathcal{G}_R$ that on R. The internal skein modules of M and M_{gl} are related as follows:*

$$\mathrm{Sk}^{int}(M_{gl})(-) \simeq \int^{V \in \mathcal{G}_{gl}} \mathrm{Sk}^{int}(M)(V \boxtimes V^* \boxtimes -) \tag{4.12}$$

Proof. Let $\mathcal{P}_R : \mathcal{G}_R \to \mathrm{SkCat}(R)$, $\mathcal{P}_{gl} : \mathcal{G}_{gl} \to \mathrm{SkCat}(\Sigma_{gl})$ and $\mathcal{P} : \mathcal{G} \to \mathrm{SkCat}(\partial M)$ denote the disk insertion functors. Note that $\mathrm{Sk}^{int}(N) = \mathrm{Sk}(N) \circ \mathcal{P}_{\partial N}$ as functors. Then

$$\begin{aligned}
\mathrm{Sk}^{int}(M_{gl}) = \mathrm{Sk}(M_{gl}) \circ \mathcal{P}_R &\simeq \int^{X \in \mathrm{SkCat}(\Sigma_{gl})} \mathrm{Sk}(M)(X \sqcup \overline{X} \sqcup \mathcal{P}_R(-)) \\
&\simeq \int^{V \in \mathcal{G}_{gl}} \mathrm{Sk}(M)(\mathcal{P}_{gl}(V) \sqcup \overline{\mathcal{P}_{gl}}(V) \sqcup \mathcal{P}_R(-)) \\
&= \int^{V \in \mathcal{G}_{gl}} \mathrm{Sk}(M)(\mathcal{P}(V \boxtimes V^* \boxtimes -) = \int^{V \in \mathcal{G}_{gl}} \mathrm{Sk}(M)^{int}(V \boxtimes V^* \boxtimes -).
\end{aligned} \tag{4.13}$$

The coend on the first line is a result of Lemma 4.6, while the equivalence in the second line follows from the fact that every X in $\mathrm{SkCat}(\Sigma_{gl})$ is isomorphic to some $\mathcal{P}_{gl}(V)$. ☐

4.2.1 Changing gates and closing punctures

For computations it is sometimes practical to change the number of gates used in the construction of internal skein algebras and modules. By *closing a gate* we will mean taking invariants of the associated disk insertion action. On the level of the internal skein algebra, this is restriction to an invariant sub-algebra generated by those skeins which do not have an endpoint at the specified gate.

By *opening a gate* we will mean passing to a larger algebra in which skeins are allowed to end at an additional gate. In practice we will often set up our marked surfaces to have sufficient gates for gluing operations, then we will close all gates near the end of a computation to arrive at the (non-internal) skein algebra.

Going further, it is sometimes necessary to puncture a surface so that it has sufficient boundary components for monadic reconstruction. In the language of [JLSS21, Sec. 1.3], this is the need for a $\mathcal{G}$-chart. Let $\Sigma^* := \Sigma \setminus \mathbb{D}$ be a stratified punctured surface. Let $\mathcal{G}_p$ denote the gates on the boundary component associated to the puncture and $\mathcal{G}_r$ the gates on the rest of the boundary. Quantum Hamiltonian reduction allows us to pass from $\mathrm{SkAlg}^{int}(\Sigma^*)$ to $\mathrm{SkAlg}^{int}(\Sigma)$. This technique is used extensively in [GJS21], where the unstratified surfaces need at most one puncture. It is a two step process In the first step we pass to the invariant sub-algebra

$$\mathrm{SkAlg}^{int}(\Sigma^*)^{\mathcal{G}_p} \subset \mathrm{SkAlg}^{int}(\Sigma^*) \tag{4.14}$$

generated by skeins which do not meet the gates at the puncture. In the second step we quotient by the relation that fixes the monodromy around the puncture. To be precise, this means setting the skein parallel to the puncture's boundary component which shares its labelling equal to the appropriate quantum dimension.

5 Parabolic induction and restriction

This section is dedicated to constructing our main example of a local coefficient system, built from the representation theory of the quantum group, its Borel subalgebra, and the universal Cartan subquotient. We refrain from giving detailed presentations of various quantum groups we consider, for which we refer the reader to [].

Let G be a reductive group, with fixed Borel subgroup $i : B \hookrightarrow G$, and its universal Cartan quotient $\pi : B \to T := B/[B,B]$.[2] Let E_i, F_i, K_i for $i = 1, \dots, n$ be the Serre generators of $U_q\mathfrak{g}$. We will denote the entries of the Cartan matrix by a_{ij}, the weight lattice by Λ, and the fundamental weights by $\lambda_1, \dots, \lambda_n$. We identify $\iota : U_q\mathfrak{b} \hookrightarrow U_q\mathfrak{g}$ with the subgroup generated by the E_i and K_i. The projection $\pi : U_q\mathfrak{b} \to U_q\mathfrak{t}$ is given by $E_i \mapsto 0$, $K_i \mapsto K_i$.

5.1 The parabolic restriction disk algebra

We will describe the disk algebra that implements parabolic restriction. The bulk of this section is a discussion of the various candidates for the central algebra associated to the defect itself. In short, there are a few options which don't work for technical reasons and two options, $\mathrm{Rep}_q B$ and $\widetilde{\mathrm{Rep}_q B} := \underline{\mathrm{End}}(\mathbb{1}) - \mathrm{mod}_{G\times\overline{T}}$, which do work and which lead to similar theories.

Let $\mathrm{Rep}_q G$ be the ribbon category of finite dimensional representations of $U_q\mathfrak{g}$. Let $\mathrm{Rep}_q T$ be the full subcategory (also ribbon) of those finite dimensional $U_q\mathfrak{t}$-representations spanned by weight vectors v_μ on which the K_i act by the ith fundamental weight $K_i \cdot v_j = q^{\langle\lambda_i,\mu\rangle} v_j$.

As mentioned, there are various categories associated to the Borel, only some of which give rise to well behaved defects between G and T regions. First, neither $U_q\mathfrak{b} - \mathrm{mod}$ nor its full subcategory of finite dimensional modules $U_q\mathfrak{b} - \mathrm{mod}^{fd}$ is well suited to the task.

A compromise is struck with $\mathrm{Rep}_q B \subset U_q\mathfrak{b} - \mathrm{mod}$, the full subcategory of *locally finite* $U_q\mathfrak{b}$-modules whose restrictions to $U_q\mathfrak{t}$ land in $\mathrm{Rep}_q T$. This was used to define the quantum decorated character stacks of [JLSS21] and is described in some detail in [JLSS21, §2.3]. It gives rise to a well behaved interface between $\mathrm{Rep}_q G$ and $\mathrm{Rep}_q T$, but as we'll see the associated skein categories do not enjoy full monadic reconstruction.

[2] T is commonly thought of as a subgroup of B, or of G. We instead treat it as a quotient because we want a $\mathrm{Rep}_q T$–module structure on $\mathrm{Rep}_q B$, instead of the other way around. We note that T defined this way as a quotient of B is canonically independent of the choice of B.

There is a $\operatorname{Rep}_q G \boxtimes \operatorname{Rep}_q T^{bop}$-action on $\operatorname{Rep}_q B$, given on objects by $(V \boxtimes \chi) \boxtimes (-) = \iota^* V \otimes \pi^* \chi \otimes (-)$. In Section 5.1.1 this will be lifted to a $(\operatorname{Rep}_q G, \operatorname{Rep}_q T)$-central structure. For now we consider what the underlying action. Let act_m be the action applied to an object $m \in \operatorname{Rep}_q B$:

$$\begin{aligned} \operatorname{act}_m : \operatorname{Rep}_q G \boxtimes \operatorname{Rep}_q T^{bop} &\to \operatorname{Rep}_q B \\ V \boxtimes \chi &\mapsto \iota^* V \otimes \pi^* \chi \otimes m. \end{aligned} \tag{5.1}$$

This has a right adjoint, act_m^R , so we have an associated monad $(A_m := \operatorname{act}_m^R \operatorname{act}_m, \mu : A_m A_m \Rightarrow A_m, \epsilon : \mathrm{id} \Rightarrow A_m)$. Next we compare the Eilenberg-Moore category of this monad with $\operatorname{Rep}_q B$ itself. There's a adjoint pair of comparison functors:

$$\operatorname{Rep}_q B \underset{\tilde{L}}{\overset{\tilde{R}}{\rightleftarrows}} A_m(\mathbb{1}) - \operatorname{mod}_{\operatorname{Rep}_q G \boxtimes \operatorname{Rep}_q T^{bop}} \tag{5.2}$$

Note that act_m^R is also denoted $\underline{\operatorname{Hom}}(m, -)$ and called the internal hom functor. Similarly, $A_m(\mathbb{1})$ is written $\underline{\operatorname{End}}(m)$ and called the internal endomorphism algebra. In the specific example of parabolic reduction $\underline{\operatorname{End}}(\mathbb{1})$ is sometimes denoted $\mathcal{O}_q(G/N)$ or $\mathcal{F}_q(N \setminus G)$. The comparison functor $\tilde{R}$ sometimes defines a reflexive embedding or even an equivalence:

Theorem 5.1 ([JLSS21, BZBJ18])**.** *Fix an action $\mathcal{A} \boxtimes \mathcal{M} \to \mathcal{M}$, where $\mathcal{A}$ is a rigid tensor category and $\mathcal{M}$ is abelian. Let m denote an object of $\mathcal{M}$, $\operatorname{act}_m : \mathcal{A} \to \mathcal{M}$ the action on m, and A_m the monad of the adjunction $(\operatorname{act}_m, \operatorname{act}_m^R)$.*

1. *Suppose that act_m^R is conservative (i.e. reflects isomorphisms.) Then we have a reflexive embedding*
$$\mathcal{M} \hookrightarrow A_m - \operatorname{mod}_{\mathcal{A}} \tag{5.3}$$
2. *Suppose that act_m^R is both conservative and colimit-preserving. Then the above embedding is an equivalence:*
$$\mathcal{M} \simeq A_m - \operatorname{mod}_{\mathcal{A}} \tag{5.4}$$

With this theorem in mind, we prove the following:

Lemma 5.2. *The functor $\operatorname{act}_{\mathbb{1}}^R : \operatorname{Rep}_q B \to \operatorname{Rep}_q G \boxtimes \operatorname{Rep}_q T^{bop}$ is conservative but not colimit preserving.*

Compare to [JLSS21, Prop. 3.41]. Hence we have a reflexive embedding

$$\operatorname{Rep}_q B \hookrightarrow \underline{\operatorname{End}}(\mathbb{1}) - \operatorname{mod}_{\operatorname{Rep}_q G \boxtimes \operatorname{Rep}_q T^{bop}} \tag{5.5}$$

Both $\operatorname{Rep}_q B$ and $\underline{\operatorname{End}}(\mathbb{1}) - \operatorname{mod}_{\operatorname{Rep}_q G \boxtimes \operatorname{Rep}_q T^{bop}}$ have $(\operatorname{Rep}_q G, \operatorname{Rep}_q^T)$-central structures.

5.1.1 The half braiding

The category $\operatorname{Rep}_q B$ can be endowed with an $(\operatorname{Rep}_q G, \operatorname{Rep}_g T)$-central structure:

$$\begin{aligned} \operatorname{Rep}_q G \otimes \operatorname{Rep}_q T^{op} &\to Z(\operatorname{Rep}_q B) \\ V \boxtimes \chi &\mapsto \big(i^*(V) \otimes \pi^*(\chi), c_{(V \boxtimes \chi, \cdot)}\big), \end{aligned}$$

where the half braiding $c_{(V \boxtimes \chi, \cdot)}$ is defined as follows. Let R^G, R^T denote the R-matrices defining the braiding for $\operatorname{Rep}_q G$ and $\operatorname{Rep}_q T$, then:

$$c_{(V \boxtimes \chi, W)} : \big(i^*(V) \otimes \pi^*(\chi)\big) \otimes W \xmapsto{\sigma_{(123)} R^G_{13} (R^T_{32})^{-1}} W \otimes \big(i^*(V) \otimes \pi^*(\chi)\big).$$

Theorem 5.3. *Let $G = \mathrm{SL}_2\,\mathbb{C}$ or $\mathrm{PSL}_2\mathbb{C}$. For any decorated surface Σ, the skein category $\operatorname{SkCat}(\Sigma)$ associated to the parabolic induction algebra is equivalent to the quantum decorated character stack $\mathcal{Z}(\Sigma)$ of [JLSS21].*

Proof. `sorry;`
This is a corollary of it being stratified factorization homology. □

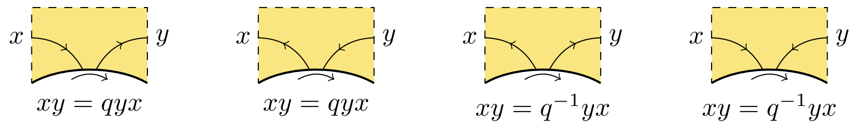

Figure 8: T-region commutation relations for skeins meeting only at a single gate.

5.2 Quantum cluster charts

5.3 Geometric interpretation

In this section we focus on the classical setting $q = 1$, we recall a number of well-known moduli spaces of connections, and we explain how they are recovered formally from the preceding constructions. A reader who is puzzled by the geometric meaning of the formal constructions – gates, disk insertions, parabolic restriction, etc. – may find the present section helpful for building some intuition.

We begin by recalling the construction of the A-polynomial [CCG+94, CL96]. The set of homomorphisms from the fundamental group of a compact manifold M to an algebraic group G is called the *G-representation variety of M*, and is denoted $R_G(M) := \mathrm{Hom}(\pi_1(M), G)$. The representation variety carries an algebraic G-action given by post-composition with the conjugation action of G on itself, and the associated GIT quotient is called the *G-character variety of M*:

$$\chi_G(M) := R_G(M) /\!\!/ G. \tag{5.6}$$

We will denote the projection $R_G(M) \to \chi_G(M)$ by π_M.

Character varieties are used to construct a polynomial knot invariant as follows. Let $K \hookrightarrow S^3$ be a knot and $n(K) \subset S^3$ an open tubular neighborhood of its image in S^3. The character variety of the complement $M_K := S^3 \setminus n(K)$ is high dimensional, singular, and in general non-reduced. To obtain a simpler object we start by considering the regular map induced by inclusion $\iota : \partial M_k \simeq \mathbb{T} \to M_k$ of the boundary torus:

$$\iota^* : \chi_{\mathrm{SL}_2}(M_k) \to \chi_{\mathrm{SL}_2}(\mathbb{T}) \tag{5.7}$$

Now we have the following chain of maps:

$$\chi_{\mathrm{SL}_2}(M_k) \xrightarrow{\iota^*} \chi_{\mathrm{SL}_2}(\mathbb{T}) \xleftarrow{\pi_{\mathbb{T}}} R_{\mathrm{SL}_2}(\mathbb{T}) \xleftarrow{\xi} R_T(\mathbb{T}), \tag{5.8}$$

where $\xi : R_T(\mathbb{T}) \to R_G(\mathbb{T})$ in induced by the inclusion $T \hookrightarrow G$. The map $\pi_{\mathbb{T}} \circ \xi$ is often replaced by an eigenvalue projection

$$\begin{array}{ccc} \chi_{\mathrm{SL}_2}(\mathbb{T}) & \xrightarrow{\qquad\quad \eta \quad\qquad} & \mathbb{C}^* \times \mathbb{C}^*/\mathbb{Z}_2 \\ (X, Y) \sim \left(\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}, \begin{pmatrix} \mu & 0 \\ 0 & \mu^{-1} \end{pmatrix} \right) & \longmapsto & (\lambda, \mu) \sim (\lambda^{-1}, \mu^{-1}). \end{array} \tag{5.9}$$

Note that η is an isomorphism, but not as well suited as $\pi_{\mathbb{T}}, \xi$ to the skein theoretic description we seek in Section 5.4.

Next, let $Y_1, \ldots, Y_n$ be the irreducible components of $\chi_{\mathrm{SL}_2}(M_K)$ with $\dim \overline{\iota^*(Y_i)} = 1$ in $\chi_{\mathrm{SL}_2}(\mathbb{T})$. The *$A$-polynomial of K* is a defining polynomial of the affine curve [3]

$$V_K := \overline{(\pi_{\mathbb{T}} \circ \xi)^{-1} \left(\iota^*(Y_1 \cup \cdots \cup Y_n) \right)} \subset \overline{R_T(\mathbb{T})}. \tag{5.10}$$

Note that $R_T(\mathbb{T}) \simeq T \times T \simeq \mathbb{C}^* \times \mathbb{C}^*$ has closure $\mathbb{C}^2$, so that the A-polynomial can be identified with a two-variable polynomial.

To obtain a specific polynomial we must fix a choice of basis, i.e. a meridian m and longitude l on the boundary torus. These determine generators ℓ, m of the coordinate ring $\mathcal{O}(\mathbb{C}^2) \simeq \mathbb{C}[\ell, m]$. We ask that the meridian bound a disk in $n(K)$ and and have linking number $+1$ with the original knot. The longitude is similarly determined by having intersection number $+1$ with the meridian. This determines $A(\ell, m)$ up to a scalar. In [CCG+94, §2.3] it's shown that this scalar can be chosen so that A has integer coefficients. We fix A up to a sign by requiring it has integer coefficients and no integer factor.

[3] This subvariety is 1-dimensional because the restriction of $\pi_{\mathbb{T}} \circ \xi$ can be understood as the quotient $(\mathbb{C}^*)^2 \to (\mathbb{C}^*)^2/\mathbb{Z}_2$ by a finite group.

Figure 9: The data specifying $\phi : R_G(\Sigma) \to \mathbb{C}$.

5.4 The skein theoretic description

A noncommutative analog of the A-polynomial, known as the A-ideal, was constructed using $\mathrm{SL}_2\,\mathbb{C}$ skein theory in [FGL01]. In [FGL01, Gel01] elements of the A-ideal were shown to annihilate the colored Jones polynomial and in some cases to determine it entirely. This relationship is now commonly understood in terms of the AJ conjecture, formulated in [Gar04] and manifested in physical terms in [Guk05].

Here we give a description of the classical A-polynomial in terms of defect skein categories with symmetric monoidal coefficients. Let

$$Cl := (\mathrm{Rep}\,G, \mathrm{Rep}\,T, \mathrm{Rep}\,G \times \mathrm{Rep}\,T^{bop} \to Z(\mathrm{Rep}\,B)) \tag{5.11}$$

be the classical coefficient system associated with parabolic induction, with $G = \mathrm{SL}_2\,\mathbb{C}$.

Let Σ_T and Σ_G denote a surface labeled by a single T or G region, respectively. When the context is clear we will also use these to denote the relevant thickened surfaces. Let $\Sigma_B := \Sigma_T \cup_B \Sigma_G$ be the stratified thickened surface with a B-defect parallel to the boundary components. We will color the knot complement M_K by G but suppress the subscript. Let $\widetilde{M_K} := M_K \cup \mathbb{T}_B$ denote the knot complement with a B-defect parallel to its boundary torus, which separates a G-labeled bulk from a small T-labeled region enclosing its boundary.

Our goal is the following equivalence of $\mathrm{SkAlg}_{Cl}(\mathbb{T}_T)$-modules:

$$\mathbb{C}[l^{\pm 1}, m^{\pm 1}]/\langle A(l,m)\rangle \simeq \mathrm{SkAlg}_{Cl}(\mathbb{T}_T) \cdot \varnothing \subset \mathrm{SkMod}_{Cl}(\widetilde{M_K}). \tag{5.12}$$

Here we've already used the fact that $\mathrm{SkAlg}_{Cl}(\mathbb{T}_T) \simeq \mathbb{C}[l^{\pm 1}, m^{\pm 1}]$.

We start by translating the various varieties and regular maps involved Section 5.3 into skein theoretic ones.

The classical (i.e. $\mathrm{Rep}\,G$) skein algebra describes functions on the G-character variety while the internal skein algebra describes those on the G-representation variety:

$$\mathcal{O}(\chi_G(\Sigma)) \simeq \mathrm{SkAlg}_{\mathrm{Rep}\,G}(\Sigma), \qquad \mathcal{O}(R_G(\Sigma)) \simeq \mathrm{SkAlg}^{int}_{\mathrm{Rep}\,G}(\Sigma^*). \tag{5.13}$$

The definition of internal skein algebra relies on our surface having a non-empty boundary. We use $\Sigma^* = \Sigma \setminus \mathbb{D}^{\sqcup n}$ to denote the appropriate punctured surface.

In (5.10), the regular map $\xi : R_T(\Sigma) \to R_G(\Sigma)$ is used to define a set map between sub-varieties $\{V \subset R_G(\Sigma)\} \to \{V \subset R_T(\Sigma)\}$. In algebraic terms, it's defining a functor $\mathcal{O}(R_G(\Sigma))$-mod $\to \mathcal{O}(R_T(\Sigma))$-mod. Looking towards monadic reconstruction of skein categories [BZBJ18, §4.1] and the appearance of presheaf-valued functors in Section 4.1, we propose that the skein theoretic manifestation of ξ is a functor $\mathrm{SkCat}_{Cl}(\Sigma_G) \to \widehat{\mathrm{SkCat}}_{Cl}(\Sigma_T)$.

Lemma 5.4. *Let $A_G := \mathcal{O}(R_G(\Sigma))$ and $A_T := \mathcal{O}(R_T(\Sigma))$ and recall that $\xi^* : A_G \to A_T$ gives A_T the structure of an (A_T, A_G)-bimodule. The functor $A_T \otimes_{A_G} - : \mathcal{O}(R_G(\Sigma))$-mod $\to \mathcal{O}(R_T(\Sigma))$-mod:*

1. *Sends a finitely generated module with support $V \subseteq R_G$ to one with support $\xi^{-1}(V) \subseteq R_T$.*
2. *Is equivalent after free completion with $\mathrm{Sk}^{int}(\mathbb{T}_B) \otimes_{\mathrm{SkAlg}^{int}_{cl}(\mathbb{T}_G)} -$.*

Proof. The first claim follows directly from $\mathrm{supp}(A_T \otimes_{A_G} M) = \xi^{-1}(\mathrm{supp}(M))$, see [Sta24, Section 056H].

Claim: $\widehat{A}_G \simeq \mathrm{SkAlg}^{int}(\Sigma)$. Our starting point for the second part of the lemma is the set of equivalences $\hat{A}_G \simeq \mathrm{SkAlg}^{int}_{\mathrm{Rep}\,G}(\Sigma^*)$ and $\hat{A}_T \simeq \mathrm{SkAlg}^{int}_{\mathrm{Rep}\,T}(\Sigma^*)$, where $\widehat{X} := \mathrm{Hom}(-, X)$.

Suppose that Σ^* has r boundary components and is genus g, so that $R_G(\Sigma) \simeq G^{2g+r-1}$.

We can build a $\mathbb{C}$-valued function ϕ on the representation variety from the following data: Pick a tuple $\rho_i : G \to \mathrm{End}(V_i)$, $i = 1, \ldots, 2g + r - 1$ of finite dimensional representations of G. For each i additionally pick a *state* $v_i \otimes f_i \in V_i \otimes V_i^*$. Define

$$\begin{aligned} \phi : G^{2g+r-1} &\longrightarrow \mathbb{C} \\ (A_1, \dots, A_{rg+r-1}) &\longmapsto \bigotimes_{i=1}^{2g+r-1} f_i(\rho_i(A_i)v_i) \end{aligned} \tag{5.14}$$

Such ϕ generate A_G, and the analogous functions generate A_T.

To get a map $A_G \to \mathrm{SkAlg}^{int}_{\mathrm{Rep}\,G}(\Sigma)$ we start by recalling (4.8):

$$\mathrm{SkAlg}^{int}(\Sigma) \simeq \int^{V \in \mathrm{Rep}\,G} \widehat{V} \otimes \mathrm{Sk}(\Sigma \times [0,1], \mathcal{P}(V)).$$

Where $\mathcal{P} : \mathrm{Rep}\,G \to \mathrm{SkCat}_{\mathrm{Rep}\,G}(\Sigma)$ is the disk insertion functor. For ϕ as in (5.14), let Γ_ϕ be the colored by the representations V_i, with states v_i, j_f as shown in Figure 9. Let $W_\phi := V_1 \otimes V_1^* \otimes \cdots \otimes V_{2g+r-1} \otimes V_{2g+r-1}^*$ We get a map $\vartheta : A_G \to \mathrm{SkAlg}^{int}_{\mathrm{Rep}\,G}(\Sigma)$:

$$\begin{array}{ccccc} A_G & \longrightarrow & \widehat{W}_\phi \otimes \mathrm{Sk}(\Sigma \times [0,1]; \mathcal{P}(W_\phi)) & \longrightarrow & \mathrm{SkAlg}^{int}_{\mathrm{Rep}\,G}(\Sigma) \\ \phi & \longmapsto & v_1 \otimes f_1 \otimes \cdots \otimes v_{2g+r-1} \otimes f_{2g+r-1} \otimes \Gamma_\phi & \longmapsto & \vartheta(\phi) \end{array} \tag{5.15}$$

where the second map follows from (4.8) and the definition of a colimit. It follows from [GJS21, Prop 2.28, Prop 2.29] that ϑ is an isomorphism when $r = 1$. The case when $r > 1$ is a small generalization.

The module structures give equivalent functors. Next we prove that $- \otimes_{\mathrm{SkAlg}^{int}_{Cl}(\mathbb{T}_G)} \mathrm{Sk}^{int}_{Cl}(\mathbb{T}_B)$ and $- \otimes_{A_G} A_T$ are equivalent functors. The idea is that ξ^* is implemented by the B-defect in $\mathbb{T}_B$.

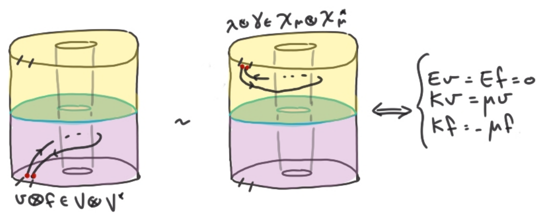

Figure 10: Moving internal skeins across the defect.

Let $\vartheta_G : A_G \to \mathrm{SkAlg}^{int}_{Cl}(\Sigma_G)$ and $\vartheta_T : A_T \to \mathrm{SkAlg}^{int}_{Cl}(\Sigma_T)$ be the isomorphisms established in the first part of the proof. Together with the eigenvalue map $\xi^* : A_G \to A_T$, this induces an $\mathrm{SkAlg}^{int}(\Sigma_G)$-action on $\mathrm{SkAlg}^{int}(\Sigma_T)$: $\Gamma_G \triangleright \Gamma_T = \vartheta_T \xi^* \vartheta_G^{-1}(\Gamma_G)\Gamma_T$. The claim is then an equivalence of $(\mathrm{SkAlg}^{int}(\Sigma_G), \mathrm{SkAlg}^{int}(\Sigma_T))$-modules Sk

□

6 Decorated skein theory and the quantum A-polynomial

The defect skein theory associated to the parabolic induction algebra is a quantization of decorated character varieties and stacks [CMR17, CMR18, JLSS21] . Inspired by this, we call it *decorated skein theory*.

We start by considering skeins that cross a defect, as in Figure.

Figure 11: The coupon studied in Lemma 6.1.

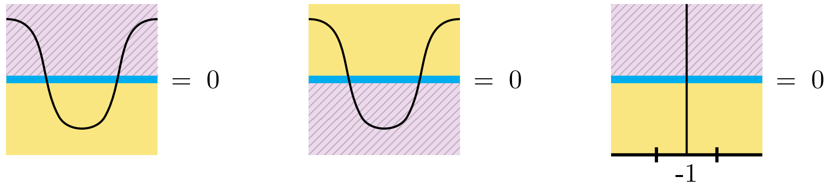

Figure 12: Some decorated skein relations implied by Lemma 6.1. We implicitly color G-region (striped purple) skeins by the fundamental representation of $\mathrm{SL}_2\,\mathbb{C}$.

Lemma 6.1. *Let V be the highest weight representation of $U_q\mathfrak{g}$ with highest weight $\overline{\lambda} = (\lambda_1, \ldots, \lambda_n)$. Similarly, let χ be the one-dimensional $U_q\mathfrak{t}$ representation with weight $\overline{\mu} = (\mu_1, \ldots, \mu_n)$. Let $H : \mathrm{Rep}_q\, G \boxtimes \mathrm{Rep}_q\, T^{bop} \to \mathrm{Rep}_q\, B$ be $\mathrm{act}_{\mathbb{1}}$, as described in Section 6.*

If V and χ have dual weights then there is a one-dimensional choice of coupons connecting them. Otherwise the zero morphism is the only possible coupon.

This lemma means that for a fixed χ and highest weight V a skein crossing the defect is either skein-equivalent to zero, or that any two coupons are skein-related by a scalar. The proof proceeds by direct computation.

Proof. The coupon in Figure (left) is colored by a morphism

$$f \in \mathrm{Hom}_{\mathrm{Rep}_q B}\big(\iota^*(V) \otimes \pi^*(\chi^*), \mathbb{1}\big). \tag{6.1}$$

First, the $U_q\mathfrak{b}$ action on $\iota^*(V) \otimes \pi^*(\chi)$ is induced by $(\iota \otimes \pi) \circ \triangle$. Let e denote the highest weight vector of V and 1 a generator of χ. Since $(\iota \otimes \pi)(\triangle(E_i)) = E_i \otimes 1$, we have $E_i \cdot (e \otimes 1) = 0$ and in general

$$E_i \cdot \left(\prod_{p=1}^{\ell} F_{j_p} e \otimes 1\right) = \begin{cases} 0 & \text{if } \sum_{p=1}^{\ell} \delta_{ij_p} = 0 \\ \left(\dfrac{\lambda_i - \lambda_i^{-1}}{q - q^{-1}}\right)^{\sum_{p=1}^{\ell} \delta_{ij_p}} \displaystyle\prod_{p=1}^{\ell} F_{j_p} e \otimes 1 & \text{otherwise.} \end{cases} \tag{6.2}$$

Since $E_1, \ldots, E_n$ all act as zero on the trivial representation, we conclude from (6.2) that

$$f\left(F_{j_1} \cdots F_{j_\ell} e \otimes 1\right) = 0 \quad \text{for} \quad 1 \leq j_1, \ldots, j_\ell \leq n. \tag{6.3}$$

Therefore f sends everything except possibly $e \otimes 1$ to zero.

The non-zero map is a $U_q\mathfrak{b}$-module morphism if and only if $f\left(K_i \cdot (e \otimes 1)\right) = K_i \cdot f(e \otimes 1)$. Since the K_i all act as one on the trivial representation and

$$K_i \cdot (e \otimes 1) = K_i \cdot e \otimes K_i \cdot 1 = \lambda_i \mu_i e \otimes 1, \tag{6.4}$$

the requirement is thus that $\lambda_i \mu_i = 1$ for $i = 1, \ldots, n$. In conclusion:

$$\mathrm{Hom}_{\mathrm{Rep}_q B}\left(\iota^*(V) \otimes \pi^*(\chi), \mathbb{1}\right) \simeq \begin{cases} \mathbb{C}_q & \text{if } \lambda_i \mu_i = 1 \text{ for } i = 1, \ldots, n \\ 0 & \text{otherwise.} \end{cases} \tag{6.5}$$

□

Note that Lemma 6.1 implies the relations shown in Figure

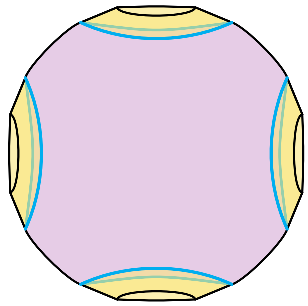

Figure 13: Our model for the boundary of a truncated decorated tetrahedron is a four-punctured sphere with three gates (not shown) per puncture. Each puncture is in an annular T-region.

6.1 Ideal tetrahedra

We will now study the main building block of Section 7.2.

Let M_{tet} denote the three ball $B^3 \subset \mathbb{R}^3$ stratified and labeled as follows. Pick four points on the boundary, and label a contractible 3-dimensional neighborhood of each by T. Label the rest of M by G. This is our model for a G-labeled tetrahedron with a small T-region surrounding each vertex. Let $\Sigma_{tet} := (\partial M_{tet}) \setminus (\mathbb{D}_T^{\sqcup 4})$ be the four-punctured sphere with an annular T-region surrounding each puncture. See Figure 13.

We equip each boundary component with three gates. This is two gates per puncture *more* than is required for Lemma 6.2, but facilitates gluing tetrahedra along faces. The 18 edges of the truncated tetrahedron determine a distinguished set $\triangle_{tet}$ of elements in $\mathrm{SkAlg}^{int}_{\mathrm{Rep}_q T^{\boxtimes 12}}(\Sigma_{0,4})$ which generate a quantum torus $\mathcal{T}_{tet}$. Two edges commute if they have no shared vertex, otherwise their exact commutation relation depends on their weights at their shared gate. In Section 7.2 we will fix an isomorphism to an abstract quantum torus $\phi : \mathcal{T}_{tet} \to \mathbb{W}_{\Omega_{tet}}$ whose product is given by $X^v X^w := q^{\frac{1}{2} v\, \Omega_{tet}\, w^{\perp}} X^{v+w}$. We will sometimes use the shorthand $X^{a_{ij}} := \phi(a_{ij}), X^{A_{ij}} := \phi(A_{ij})$ for generators but emphasize that in general $\phi(b) = q^{\lambda_b} X^b$ for some non-zero power λ_b.

The skew symmetric matrix Ω_{tet} controlling the product is obtained from considering the commutation relations in Figure 8 as applied to the generators shown and given a total order in Figure 14.

$$\Omega_{tet} := \begin{pmatrix} \Omega_{sh} & & & & \\ & \Omega_{sh} & & & \Omega_{sh,lg} \\ & & \Omega_{sh} & & \\ & & & \Omega_{sh} & \\ & -\Omega_{sh,lg}^{\perp} & & & \end{pmatrix}, \Omega_{sh,lg} := \begin{pmatrix} 0&1&1&0&0&0\\ 1&0&1&0&0&0\\ 1&1&0&0&0&0\\ 0&0&0&1&1&0\\ 1&0&0&1&0&0\\ 1&0&0&0&1&0\\ 0&0&0&1&0&1\\ 0&1&0&0&0&1\\ 0&1&0&1&0&0\\ 0&0&0&0&1&1\\ 0&0&1&0&1&0\\ 0&0&1&0&0&1 \end{pmatrix} \tag{6.6}$$

and $\Omega_{sh} := \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}$.

The following result appears in essence as [JLSS21, Prop 4.11]

Lemma 6.2. *The Ore localization of the internal skein algebra at $\triangle_{tet}$ is isomorphic to $\mathcal{T}_{tet}$:*

$$\mathrm{SkAlg}^{int}_{GB_T}(\Sigma_{0,4})[\triangle^{-1}] \simeq \mathcal{T}_{tet}. \tag{6.7}$$

Lemma 6.2 reduces to [Mul16, Thm 6.14], after noting that the stated skeins of [Mul16] are isomorphic to our internal skein algebras by [Haï22]. The other main difference is that unlike in Muller's theorem, $\triangle$ is not a full triangulation of $\Sigma_{0,4}$, since we could add additional non-parallel non-intersecting long edges.

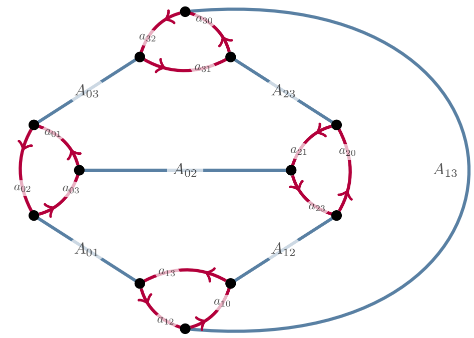

Figure 14: The full set of short and long edges which generate $\mathcal{T}_\triangle$. The long (blue) edges have weight +1 at each gate and pass between T-regions. The short (red) edges are oriented to go from their weight +1 end to their weight -1 end and remain in a single T region. For the sake of matrix computations, the generators are ordered as follows: $a_{01}, a_{02}, a_{03}, a_{10}, a_{13}, a_{12}, a_{20}, a_{21}, a_{23}, a_{30}, a_{32}, a_{31}, A_{01}, A_{02}, A_{03}, A_{12}, A_{13}, A_{23}$.

However, any two skeins incident to different gates on the same T-region puncture are related by a product of short edges. Therefore the missing edges are both invertible in the localization and contained in $\mathcal{T}_\triangle$.

To power our computations in Section 7, we now write the skein module of M_{tet} as a quotient of quantum torus $\mathcal{T}_{tet}$. We use the shorthand $\mathrm{SkAlg}(\Sigma)[\triangle^{-1}] := \left(\mathrm{SkAlg}^{int}(\Sigma)[\triangle^{-1}]\right)^T$ and similarly for skein modules.

Lemma 6.3. *The localized submodule generated by the empty skein* $\mathrm{SkAlg}(\Sigma_{tet})[\triangle^{-1}]\cdot\varnothing \subset \mathrm{SkMod}(M_{tet})[\triangle^{-1}]$ *is isomorphic to the T-invariant subalgebra of $\mathcal{T}_\triangle$ quotiented by the left ideal generated by the bulk relation*

$$q^{5/2}(A_{03}a_{32}a_{01})(A_{12}a_{23}a_{10}) + q^{3/2}(A_{01}a_{03}^{-1}a_{12}^{-1})(A_{23}a_{21}^{-1}a_{30}^{-1}) + q^{3/2}A_{13}A_{02}, \tag{6.8}$$

together with the puncture monodromy relations:

$$\begin{aligned} &q^{-3/2}a_{01}a_{02}a_{03} + 1 && q^{-3/2}a_{10}a_{12}a_{13} + 1 \\ &q^{-3/2}a_{20}a_{21}a_{23} + 1 && q^{-3/2}a_{30}a_{31}a_{32} + 1. \end{aligned} \tag{6.9}$$

Where the variables are as shown in Figure 14.

Note that the short edges in the crossing relation ensure that the three terms have the same T-weights. Restricting to the submodule generated by the empty skein corresponds to the restriction to the image of ι^* in (5.10).

Proof. By assumption the module is cyclic, and by Lemma 6.2,

$$\mathrm{SkAlg}(\Sigma)[\triangle^{-1}] \simeq \mathcal{T}_\triangle^T. \tag{6.10}$$

Hence $\mathrm{SkAlg}(\Sigma_{tet})[\triangle^{-1}]\cdot\varnothing \simeq \mathcal{T}^T_{\triangle_{tet}}/I$ for some left ideal I. The relations (6.9) are imposed by closing the four punctures after taking invariants. Let γ_i be the closed simple curve surrounding the ith puncture and α_i the associated skein, note that the α_i are exactly the monomials in (6.9). In closing each puncture we identify α_i with $-1 = \langle\theta_T\rangle \dim\chi = -q^{-1}q$, the trace of the twist in Rep_q^T. See Figure 15.

Let $\mathbb{D}_4$ be the disk with four disjoint contractable T-regions along its boundary, so that $\mathbb{D}_4 \times I \simeq M_{tet}$. We consider a single gate in each T-region, labeled $0, 1, 2, 3$ in clockwise order in both $\mathbb{D}_4$ and M_{tet}, and note that

$$\mathrm{Sk}^{int}(M_{tet}) \simeq \mathrm{Sk}^{int}(\mathbb{D}_4 \times I) \simeq \mathrm{SkAlg}^{int}(\mathbb{D}_4)\cdot\varnothing_{\mathbb{D}_4\times I} \tag{6.11}$$

as left $\mathrm{SkAlg}^{int}(\mathbb{D}_4)$ modules[4]

[4] The action on M_{tet} is induced by the embedding $\mathbb{D}_4 \hookrightarrow \Sigma_{tet}$.

Figure 15: The framing on B_{13} versus A'_{13} (left) and in the puncture monodromy (right).

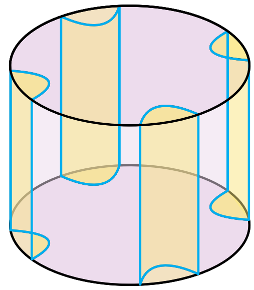

Figure 16: The identity cobordism of $\mathbb{D}_4$ used in the proof of Lemma 6.3.

Let $\triangle_{02} := \{B_{01}, B_{02}, B_{03}, B_{12}, B_{23}\} \subset \mathrm{SkAlg}^{int}_{\mathrm{Rep}_q T^4}(\mathbb{D}_4)$ be a triangulation of the quadrilateral. By Lemma 6.2, $\mathrm{SkAlg}^{int}_{\mathrm{Rep}_q T^4}(\mathbb{D}_4 \times I)[\triangle_{02}^{-1}]$ is generated by $\triangle_{02} \cup \triangle_{02}^{-1}$, and $B_{13} \sim B_{02}^{-1}\left(q^{1/2} B_{03} B_{12} + q^{-1/2} B_{01} B_{23}\right)$ in the localization.

To get the exact form of (6.8), we identify the four-gate localized internal skein module of M_{tet} with the sub-quantum torus of $\mathcal{T}_{\triangle_{tet}}$ generated by

$$\begin{aligned} A'_{01} &:= q A_{01} a_{03}^{-1} a_{12}^{-1} \quad & A'_{02} &:= A_{02} \quad & A'_{03} &:= q A_{03} a_{32} a_{01} \\ A'_{12} &:= q A_{12} a_{23} a_{10} \quad & A'_{13} &:= A_{13} \quad & A'_{23} &:= q A_{23} a_{21}^{-1} a_{30}^{-1} \end{aligned} \tag{6.12}$$

and note that the isomorphism between M_{tet} and $\mathbb{D}_4 \times I$ is given by

$$A'_{ij} \mapsto \begin{cases} -q^{3/2} B_{13} & ij = 13 \\ B_{ij} & \text{otherwise.} \end{cases} \tag{6.13}$$

where A'_{13} picks up a sign due to a framing difference, see Figure 15.

□

7 Quantizing the A-polynomial

7.1 Quantum tori and central relations

In this section we discuss some operations on quantum tori which can be realized on the level of their underlying lattices. These methods are elementary but significantly reduce the complexity of the procedures outlined in Section 7.2

Definition 7.1. *Let Λ be a lattice with a skew symmetric pairing $\langle -, - \rangle : \Lambda \times \Lambda \to \mathbb{Z}$. The* quantum torus *$\mathbb{W}_\Lambda$ is the $\mathbb{C}_q := \mathbb{C}[q^{\frac{1}{2}}, q^{-\frac{1}{2}}]$ algebra generated by X^v, $v \in \Lambda$ with multiplication given by*

$$X^v X^w = q^{\frac{1}{2}\langle v, w\rangle} X^{v+w} \tag{7.1}$$

Recall that a sublattice $\Gamma \subset \Lambda$ is called co-isotropic if the pairing restricted to Γ vanishes identically; this is equivalent to asking that the subalgebra generated by X^v for $v \in \Gamma$ is commutative. A much stronger condition on Γ is that it lie in the kernel of the pairing; this is equivalent to asking that the subalgebra generated by X^v for $v \in \Gamma$ is central. Finally, we say that Γ is unimodular if the quotient Λ/Γ is torsion-free.

We require the following elementary lemma, whose proof follows easily from the fact that $\mathbb{W}_\Lambda$ is free over $\mathbb{W}_\Gamma$ with basis given by monomials X^v for $v \in \Lambda/\Gamma$.

Lemma 7.2. *Let $\Gamma \subset \Lambda$ be a co-isotropic sublattice, and fix a character $\chi : \Gamma \to \mathbb{C}$. Then a basis for the induced module $\mathbb{W}_\Lambda \otimes_{\mathbb{W}_\Gamma} \mathbb{C}_\chi$ is given by $\{X^v,$ for $v \in \Lambda/\Gamma\}$. In particular, suppose that Γ is unimodular and lies in the kernel of the pairing. Then $\mathbb{W}_\Lambda \otimes_{\mathbb{W}_\Gamma} \mathbb{C}_\chi$ is naturally an algebra, and we have an isomorphism,*

$$\mathbb{W}_\Lambda \otimes_{\mathbb{W}_\Gamma} \mathbb{C}_\chi \cong \mathbb{W}_{\Lambda/\Gamma}.$$

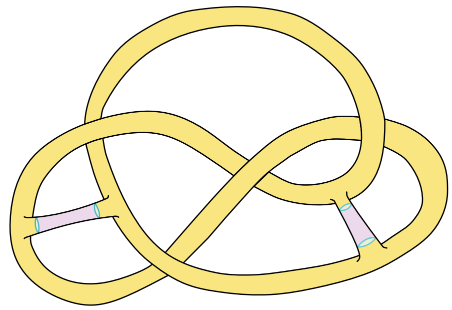

Figure 17: The full decorated surface Σ_t which appears in the computation of the quantum A-polynomial. It is drawn here for a two-tetrahedra triangulation of the 4_1-knot complement. For any knot, Σ_t will be a T-region torus with a G-region handle for each edge of the original ideal triangulation.

Remark 7.3. *The relative tensor product* $\mathbb{W}_\Lambda \otimes_{\mathbb{W}_\Gamma} \mathbb{C}_\chi$ *may be presented as the quotient by the left ideal* $I = \langle X^{v_1} - \chi(v_1), \ldots, X^{v_k} - \chi(v_k)\rangle$*, for any spanning set* $v_1, \ldots, v_k$ *of* Γ*. This is how it typically arises.*

7.2 Computations

In this section we describe in detail the computation of the quantum A-polynomial. Computations were implementation in the mathematics software system SageMath [The22] and rely on Singular [DGPS24] for constructing non-commutative Gröbner basis.

Let $\widetilde{M}_K = M_k \cup_{\mathbb{T}_G} \mathbb{T}_B$ denote the knot complement with a B-defect separating a T-region tubular neighborhood of its boundary from a G-region bulk. Our goal is to compute the cyclic $\mathrm{SkAlg}(\mathbb{T}_T)$-submodule

$$S_K := \mathrm{SkAlg}(\mathbb{T}_T) \cdot \varnothing \subset \mathrm{SkMod}(\widetilde{M}_K) \tag{7.2}$$

generated by the empty skein.

Fix an ideal triangulation of the knot exterior, with t tetrahedra. There is a corresponding decomposition of $\widetilde{M}_K$ into the decorated tetrahedra described in Section 6.1. Faces are glued together via stratified boundary connect sum along copies of $\mathbb{D}_3$, leading to a genus $t+1$ decorated surface Σ_t shown in Figure 17.

Our goal is an expression

$$\mathrm{SkMod}(\widetilde{M}_K) \supseteq \mathrm{SkAlg}(\mathbb{T}_T) \cdot \varnothing \simeq \mathrm{SkAlg}(\mathbb{T}_T)/\langle A_q\rangle \tag{7.3}$$

such that specializing $q^{1/2} \to -1$ recovers the classical A-polynomial. Since our computations rely heavily on quantum cluster charts and therefore localizations, we instead produce an expression for the empty submodule of a localization $\mathrm{SkMod}(\widetilde{M}_K)[\triangle_K^{-1}]$, which is a quotient of the skein algebra by an ideal generated by a multiple of the quantum A-polynomial. We emphasize that this is a side effect of the computational methods, and not the decorated skein theory.

Threads and gluing relations The faces of the t decorated tetrahedra which triangulate $\widetilde{M}_K$ are copies of $\mathbb{D}_3$. When the faces are glued together the result is a decorated genus $t+1$ surface as shown in Figure 17. This gluing operation introduces additional skeins called *threads*. The edges $\triangle_{tet}$ of each truncated tetrahedron's boundary triangulation together with these threads generate a rank $30t$ quantum torus. A quantum cluster chart of the internal skein algebra of the associated decorated surface is then obtained by taking the quotient by the *gluing relations* shown in Figure 19. Each long edge will appear in two relations, while each short edge will appear in one, leading to a total of $12t$ gluing relations. These relations are all central and the coordinates of their monomials span a direct summand of the rank $30t$ lattice generated by all short edges, long edges, and threads. We can therefore apply Lemma **??**

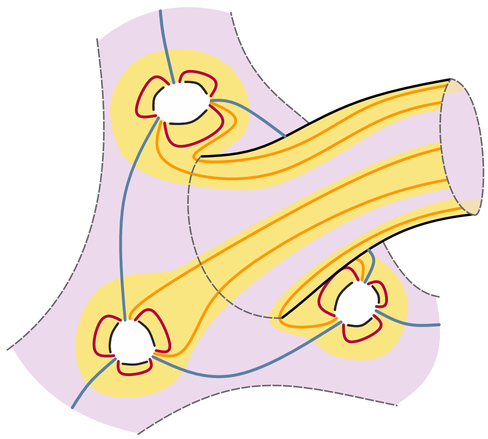

Figure 18: A portion of a surface with part of the gluing handle attachment shown. Long edges (blue) connect punctures, short edges (red) surround punctures, and threads (orange) travel between tetrahedra.

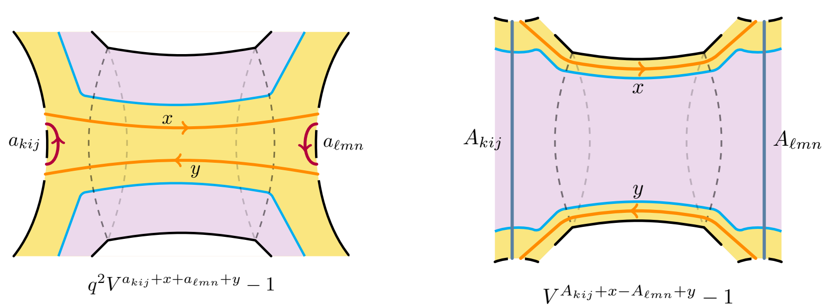

Figure 19: The gluing relations which identify pairs of short (left) and long (right) edges. These rely on the addition of threads.

Invariants and central relations Next we restrict to the T-invariant subalgebra of the aforementioned rank $30t$ quantum torus, thereby closing all gates, and quotient by the gluing relations of Figure 19 and the puncture monodromy relations of (6.9). These relations are all central in the T-invariant subalgebra, and so the quotient can therefore be computed on the level of the underlying lattice, see Section 7.1.

Elimination, specialization, and A_q We have now arrived at a quantum cluster chart for the internal skein algebra of the decorated surface shown in Figure 17. We construct a basis for this chart consisting of the longitude, meridian, *thread monodromy*, and extraneous variables. In this algebra we form the left ideal generated by the bulk relations (6.8) and use a noncommutative Gröbner basis to eliminate all dummy variables. The longitude and meridian will always commute with the thread monodromy relations, which correspond to curves in the T-region which can be written only in terms of the threads. To obtain the skein algebra of the origianl T-colored boundary torus of $\widetilde{M}_K$ and the skein subodule generated by the empty skein, we finally specialize these thread monodromy variables to their appropriate q-powers.

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